A Generalized Notion of Time for Modelling Temporal Networks

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Abstract: Most approaches for modelling and analysing temporal networks do not explicitly discuss the underlying notion of time. In this paper, we therefore introduce a generalized notion of time for temporal networks. Our approach also allows for considering non-deterministic time and incomplete data, two issues that are often found when analysing independent samples extracted from online social networks, or when trying to make predictions for different possible future scenarios, for example. In order to demonstrate the consequences of our generalized notion of time, we also discuss the implications for the computation of (shortest) temporal paths in temporal networks.

1 Introduction

Temporal networks are a means for modelling and analysing the temporal dimension of complex (networked) systems. However, most of the literature on temporal networks either does not explicitly discuss the underlying notion of time or uses a rather restricted conception of time. In this paper, we discuss a generalized notion of time that also allows for possible deviations from a linear flow of time. Thereby, our approach allows for considering non-determinism and incomplete data when analysing temporal networks (two issues that often appear when either dealing with simulations or with real-world data, such as different samples extracted from the same online social network).

In particular, we use a variant of the multilayer network concept to construct temporal networks alongside our generalized notion of time. That is, we consider a temporal network, as a (temporal) sequence of networks, a notion that can easily be expressed via the well-known multilayer network concept. This approach allows to clearly separate time from other (temporally varying) attributes that are attached to the edges or vertices in a network.

In order to demonstrate the consequences of our generalized notion of time, we also discuss the implications for the computation of (shortest) paths in temporal networks. In this context, we briefly sketch how the novel interpretation of temporal paths affects path-based centrality measures, such as a temporal betweenness centrality.

1.1 Basic Definitions

The notion of multilayer networks was introduced to unify several independently developed graph-like structures such as temporal networks, networks with multiple types of connections, or interdependent networks, into a single approach (Kivelä and Porter, 2018). Roughly, two types of approaches exist. The first one, understands multilayer networks as a family of graphs connected with inter-graph edges and is used in (Boccaletti et al., 2014), (Solá et al., 2013) (Wang et al., 2017), (Spatocco et al., 2018) and (Fenu and Higham, 2017). The second one makes the layers of a multilayer network explicit by defining vertices as a tuple representing their layer membership. This approach appears in publications such as (Kivelä and Porter, 2018), (Kivelä et al., 2014), (Tomasini, 2015) and (Cozzo et al., 2016). Both definitions can be represented as a tensor, existing in an analogue to the adjacency matrix of ordinary graphs. Applications and descriptions of such a tensor formalism can be found in (De Domenico et al., 2013), (Solé-Ribalta et al., 2016) and (Aleta and Moreno, 2018), for example.

Technically we rely on the first line of thought towards multilayer networks, i.e. the conception of multilayer networks as a family of graphs. However, since in this paper we are dealing with weighted multilayer networks, a slight adaptation is required to make those weights explicit. Leading us to Definition 1.1.

Definition 1.1. *A weighted multilayer network is a triple $\mathcal{M} := \langle \mathcal{G}, \mathcal{E}, \omega \rangle$ such that for some arbitrary set of labels I*

- $\mathcal{G} := (G_\alpha)_{\alpha \in I}$ *is a family of weighted graphs* $G_\alpha := (V_\alpha, E_\alpha, \omega_\alpha)$ *such that* $\forall \alpha, \beta \in I\ \alpha \neq \beta$ *we have* $V_\alpha \cap V_\beta = \varnothing$;
- $\mathcal{R} := (R_{\alpha\beta})_{\alpha,\beta \in I}$ *is a family of relations, such that* $\forall \alpha, \beta \in I\ \alpha \neq \beta$ *we have* $R_{\alpha\beta} \subseteq V(G_\alpha) \times V(G_\beta)$;
- $\omega : I \times I \times E(\mathcal{M}) \to \mathbb{R}$ *is the function*

$$\omega_{\alpha\beta} := \begin{cases} \omega_\alpha(e) & \alpha = \beta \\ \mathbb{R} & otw. \end{cases}$$

We define $\mathcal{V}(\mathcal{G}) := \{V(G) \mid G \in \mathcal{G}\}$ *and* $V(\mathcal{G}) := \bigcup_{V \in \mathcal{V}(\mathcal{G})} V$ *and . As well as* $\mathcal{E}(\mathcal{G}) := \{E(G) \mid G \in \mathcal{G}\}$ *and* $E(\mathcal{G}) := \bigcup_{E \in \mathcal{E}(\mathcal{G})} E$. *Moreover, let* $E(\mathcal{R}) = \bigcup_{R \in \mathcal{R}} R$. *Lastly,* $V(\mathcal{M}) := V(\mathcal{G})$ *and* $E(\mathcal{M}) := E(R) \cup E(\mathcal{G})$.

Informally, a multiplex network (or simply "multiplex" for short) can be understood as an edge coloured network, or a network consisting of a fixed set of vertices connected by different types of edges. Alternatively, it can be understood as a multilayer network where all layers share the same set of vertices, i.e. $\forall V, W \in V(\mathcal{G})\ V = W$ (Kivelä et al., 2014; Boccaletti et al., 2014). For the purposes of this paper, we reject the strong equality implied by those characterisations, since strong equality is not well-suited for our purposes. As for example, when analyzing networks with respect to time one can make the small semantic distinction, that an object x at time t may not share the same properties as the same object x at time $t' \neq t$.

Definition 1.2. *Let* $\mathcal{M} := \langle \mathcal{G}, \mathcal{R}, \equiv \rangle$ *be a multilayer network with equivalence such that* $\forall \alpha \forall v, w \in V(G_\alpha)\ v \equiv w \implies v = w$. *Moreover, The equivalence classes generated by* $\equiv$ *are denoted as* $\overline{v} := \{w \mid w \in V(\mathcal{G})\ v \equiv w\}$ *and we write* $\overline{v}_\alpha := v \in V_\alpha \cap \overline{v}$.

Often the property $\exists R \subseteq E(\mathcal{R})$ such that $\equiv$ is the symmetric, reflexive and transitive closure of R is highly desirable (see Definition 1.2). However, in some cases e.g. branching models of time, this property is too restrictive.

Definition 1.3. *Let* $\mathcal{M} := \langle \mathcal{G}, \mathcal{R}, \equiv \rangle$ *be a multilayer network with equivalence. It is called a multiplex iff* $\forall v \in V(\mathcal{G})\ \overline{v} = |\mathcal{G}|$.

From this it follows that for every v in every layer there is w such that $v \equiv w$. Unfortunately, when modelling the progression of time, equivalence is necessary but not sufficient. That is, for achieving the required expressiveness we have to impose an additional order onto our structure.

Definition 1.4. *Let* $\mathcal{M} := \langle \mathcal{G}, \mathcal{R}, \equiv, \leq \rangle$ *be a multilayer network with equivalence and order. If* $\leq$ *is a partial ordering on* $\mathcal{G}$. *Moreover,* $G_\alpha \leq G_\beta \iff (\forall v \in G_\alpha \forall w \in G_\beta (v \equiv w \implies v \leq w) \wedge (v \not\equiv w \implies v \nparallel w))$

2 Generalized Flow of Time

First, we abstract from the notion of a temporal network to discuss more general observations regarding objects in a temporal dimension. When studying the notion of time we distinguish between two classes of properties. Informally, the class of properties concerned with the density of time and the one determining the structure of time. Many concepts mentioned in this section can be found in and/or build on the ideas presented in (Venema, 1998), (Markosian et al., 2016), (Goranko and Galton, 2015) and (Burgess, 1979).

Definition 2.1. *Let* $\Phi := \langle T, R, \nu \rangle$ *be a structure representing the flow of time, where T is a set of points, R is a relation over T, i.e.* $R \subset T \times T$ *and* $\nu : T \to \wp(\mathcal{L})$ *is a function assigning each point in time a set of properties P described in some language* $\mathcal{L}$. *If the properties at a point in time are not relevant we write* $\Phi := \langle T, R \rangle$.

The function ν is best understood as a function that assigns each point in time a description of the world. R simply puts the points in time into relation. For example, one can embed a graph into a point in time, by attaching some axiomatisation of the desired graph theory together with a characterisation of the graph itself. There are some interresting conceptions of how points in time may relate, thus in its most gernral form R should be open to interpretation. For example, in (Taylor et al., 2017) time was conceived as an undirected, discrete flow. However, as we intend to discuss a class of specific notions of time, allowing an arbitrary relation R is cumbersome. Hence, we restrict ourself for all subsequent discussions to the set of structures building upon a directed notion of time.

Definition 2.2. *(Venema, 1998). Let* $\Phi := \langle T, \leq \rangle$ *be a structure representing the flow of time, where T is a set of points and* $\leq$ *is a partial order over T.*

For more on partially orders see (Matthews, 1994). Moreover, we write $x < y := x \leq y \wedge x \neq y$ as a shorthand. Given this directed notion of time we, will not investigate cyclic models of time. As otherwise given $x \leq y \wedge y \leq z \wedge z \leq x$ and transitivity the whole time line collapses into a single point, i.e. $x = y = z$.

By imposing further restriction onto $\leq$ we can develop certain notions of time. For example, a flow of time is *linear* if it is total, i.e. $\forall x, y \in T\ x \leq y \vee y \leq x$; it is *strictly linear* if it is total and strict, i.e. $\forall x, y \in$

$T\ x \prec y \lor y \prec x \lor x = y$; it is *backwards-branching* if for a point s marking the present there are two incomparable points in the past i.e. $\exists x,y \in T\ x \preceq s \land y \preceq s \land x \nparallel y$; it is *forwards-branching* if for a point s marking the present there are two incomparable points in the future i.e $\forall x,y \in T\ x \prec y \lor y \prec x \lor x = y$; it is *backwards-serial* if there is always another point in the past $\forall x \exists y \in T \in T\ y \prec x$; and it is *forwards-serial* if there is always another point in the future $\forall x \exists y \in T \in T\ x \prec y$ (Venema, 1998).

The concept of linear time is a highly intuitive understanding of time. Time flows within a straight line. There are no alternative time lines, no branching and no cycles, allowing us to work in a deterministic fashion. For example, one encounters this notion of time when dealing with ordinary time series. Moreover, a good part of the literature regarding temporal networks, is concerned with this analytical framework. However, one can easily conceptualise scenarios, where we deal with some kind of uncertainty or non-determanism. Here the notion of possible worlds can guide our reasoning. Kripke models, which provide the foundation for the semantics of modal logic, heavily rely on the concept of possible worlds (Venema, 1998; Van Ditmarsch et al., 2007).

Thus, each branch represents a possible future, i.e. expressing the non-determinism in the future. For example, having a linear flow of time in the past that branches into the future, expresses that we are certain what happend in the past, but we can not predict the future with certainty. How those futures are obtained precicely, may it be through statistical inference, by consulting experts with domain knowledge or being a product of a simulation with randomness is currently not of our concern. Considering possible worlds is especially useful when dealing with discrete objects such as graphs. That is, rather than introducing fuzzy edges (Sunitha and Mathew, 2013) we can, at least for our purposes, consider several worlds where an edge exists and some in which it does not. Similarly, backwards-branching could encode the notion of unreliable data into our models. For example, if there are two contradicting measurements of the same phenomena concerning the same instance, one expresses them as two incomparable elements within the flow of time.

Moreover, regardless of forward- or backward-branching, we do allow for collapsing flows of time. That is, two branches could meet at some point of time. Recall the unreliable data example, as an example of the applicability of such a structure. Before discussing the "density" of the flow of time, a remark about infinity. While infinity is interesting from a theoretical perspective, we will work within a finite world. That is, at least in the context of (real-world) network analysis, we are dealing with a finite stream of finite data. Hence, we always assume that we have only a finite number branches and that every branch has a start and/or an endpoint. Therefore, forward-serial and backward-serial flows of time will not be discussed further.

When discussing the density of time, three notions are common. Firstly, a flow of time is *discrete* if $\forall x,y(x \preceq y \to \exists z(x \preceq z \land \neg \exists u(x \preceq u \land u \preceq z)))$, i.e. if x is neither the start or the end of the interval then there must exists a subsequent point in time z, such there are no other instances of time in-between. Hence, for every non-final instant x there exists a direct successor y and can best be understood in analogy to the integers $\mathbb{Z}$. It is *dense* if $\forall x,y(x \preceq y \to \exists z(x \preceq z \land z \preceq y))$, i.e. if x is neither the start or the end of the interval then for any other point in time y, there must be another instance z after x and before y and thus can be taken in analogy to the rational numbers $\mathbb{Q}$. Lastly, it could also be continuous, i.e. while density is required it does not imply continuity (consider $\sqrt{2}$). Hence, this can be taken in analogy to $\mathbb{R}$. However, as most measurements of the real world are processed by inherently discrete machines and this paper is geared towards the analysis of discrete sequences of discrete objects, we will limit ourself to a discrete conception of time. However, for analytical and predictive purposes the other two models of time, especially the continuous one, should be investigated further in the context of temporal networks (Venema, 1998; Dedekind and Beman, 1901; Burgess, 1979).

Having a concrete conception of a successor, i.e. $S(a,b) := \forall x \in T\ x \prec a \lor b \prec x$ holds. We can formalise steps in time. However, while in linear time the predicate S corresponds to bijective mapping s. As soon as branching is permitted, $S(a,b)$ can be satisfied by multiple elements.

Definition 2.3. *Let $\Phi := \langle T, \preceq \rangle$ be a flow of time, then $s(t)$ is called the successor function iff*

$$s : T \to \wp(T) \quad s(x) \mapsto \{y \mid y \in T\ S(x,y)\}$$

for the successor predicate $S(a,b)$. Moreover, $p(x) := \{y \mid \forall y \in T\ x \in s(y)\}$ is called the predecessor function

By taking several steps we obtain a path in time.

Definition 2.4. *Let $\Phi := \langle T, \preceq \rangle$ be a flow of time, then*

- *a forward path in time $a \rightsquigarrow^{+} b$ between a and b is a sequence $a \rightsquigarrow^{+} b := a, x_2, \ldots x_{n-1}, b$ where $x_{i+1} \in s(x_i)$;*
- *a backwards path in time $a \rightsquigarrow^{-} b$ between a and b is a sequence $a \rightsquigarrow^{-} b := a, x_2, \ldots x_{n-1}, b$ where $x_{i+1} \in p(x_i)$;*

- *if $a \leq b$ a path in time is $a \rightsquigarrow b := a \rightsquigarrow^+ b$ and if $b \leq a$ a path in time is $a \rightsquigarrow b := a \rightsquigarrow^- b$.*

The size of a path in time is determined by the elements in the path and denoted as $|a \rightsquigarrow b|$ and can be understood as the amount of steps in time between a and b in $a \rightsquigarrow b$.

Notice, if $a \rightsquigarrow^- b$ there exists some $b \rightsquigarrow^+ a$ such that $a \rightsquigarrow^- b = b \rightsquigarrow^+ a$. Moreover, if $a \rightsquigarrow b$ and $b \rightsquigarrow z$ for $a \leq b$ and $z \leq b$, the concatenation of those two paths $a \rightsquigarrow b \rightsquigarrow z$ is no path in time, since it is neither a forwards, nor a backwards path in time. Hence, by preventing a path from changing directions there can not be a path between two branches. The concept of size can be generalised to obtain a notion of length.

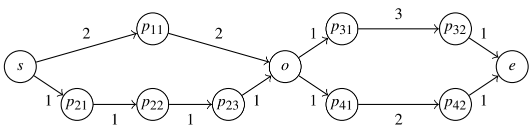
Definition 2.5. *Let $\Phi := \langle T, \leq, \omega \rangle$ be a flow of time with spacing, with the function*

$$\omega : \mathcal{S}(T) \to \mathbb{R}^+,\ x \mapsto \omega(x)$$

Where $\mathcal{S}(T) := \{(x,y) \mid \forall x,y \in T\ S(x,y)\}$ contains all steps in time. The length of a path in time $x \rightsquigarrow y$ with respect to ω is thus $|x \rightsquigarrow y|_\omega := \sum_{(p,s(p)) \in E(x \rightsquigarrow y)} \omega(p, s(p))$.

Intuitively, ω stretches time, i.e. it assigns a duration onto a step in time. However, while the length of a path in time is a useful concept, it can not serve as a measure of distance.

Example 2.6. *Consider the flow of time $\Phi := \langle T, \leq, \omega \rangle$*



What is the distance δ between s,o and between o,e? For the prior when considering $|\cdot|$ we have $d(s,o) = 1$ or $d(s,o) = 4$ and for $|\cdot|_\omega$ we have $d_\omega(s,o) = 4$ or $d_\omega(s,o) = 4$. In analogue, we have $d(o,e) = 2$ or $d(o,e) = 2$ and $d_\omega(o,e) = 5$ or $d_\omega(o,e) = 4$.

To resolve this ambiguity, we make use of a quasi-metric with infinity. A fairly natural choice for modelling non-cyclic, directed flows of time. As one can easily travel into the future one instance at a time, but travelling into the past is (most likely) impossible. Moreover, by assigning an infinite distance to backtracking within our flow of time, the distance between two chains, becomes also infinite. One can easily check, that this holds for chains with and without a join. More information about quasi-metrics can be found in (Matthews, 1994; Schroeder, 2006).

For the remainder of this paper we use the following quasi-metric.

Definition 2.7. *Let $\Phi := \langle T, \leq, \omega \rangle$ be a flow of time with spacing. We call*

$$\delta_\omega(a,b) := \begin{cases} \min\limits_{a \rightsquigarrow b}(|a \rightsquigarrow b|_\omega) & a \leq b \\ \infty & otw. \end{cases}$$

the distance between a and b (with respect to ω). Moreover, we call δ_1 the step distance between a and b.

Proposition 2.8. *$\Phi := \langle T, \leq \rangle$ and $\Phi := \langle T, \leq, \omega \rangle$ are quasi-metric flows of time.*

Proof. Since δ_1 is a special case of δ_ω we only consider the latter. Firstly, $\delta_\omega(x,y) \geq 0$ holds as $\forall (x,y) \in S(T)\ \omega(x,y) \in \mathbb{R}^+$. Secondly, $\delta_\omega(x,y) = 0 = \delta_\omega(y,x) \iff x = y$ hold because $x = y \iff |x \rightsquigarrow y| = 0$, especially with respect to ω, and the empty sum is always 0. Finally, $\delta_\omega(x,z) \leq \delta_\omega(x,y) + \delta_\omega(y,z)$. Firstly, if $z < x$ then $\delta_\omega(x,z) = \infty$. Since there are no circles, it must be that either $\delta_\omega(x,y) = \infty$ or $\delta_\omega(y,z) = \infty$. For $y < x$ or $z < y$ we have that in either case backtracking would be required, leading to a shift in direction, which by definition is not a valid path in time. Lastly, $x < y < z$. Assume the contrary. Thus $\delta_\omega(x,z) > \delta_\omega(x,y) + \delta_\omega(y,z)$. Hence, $\exists x \rightsquigarrow y, y \rightsquigarrow z$, resulting in $x \rightsquigarrow y \rightsquigarrow z$ such that $|x \rightsquigarrow z|_\omega > |x \rightsquigarrow y \rightsquigarrow z|_\omega$. But $|x \rightsquigarrow z|_\omega$ is minimal. □

However, while this may allow us to speak about distance, in one form or another. It creates some semantic inconveniences or inconsistencies. Some of which are addressed in the following subsection.

2.1 Linear and Homogeneous Flows

The notion of distance developed above may lead to unfortunate outcomes (see Example 2.9). Many of which can be reduced to the fact that, in their general form, flows in time can easily be used to express time as having varying density, i.e. the spacing between points may vary within a single chain or across branches. While sometimes useful, e.g. consider relativity of time, a characterisation of time without such properties is equally desirable.

Example 2.9. *Consider the flow of time $\Phi := \langle T, \leq \rangle$ as in Example 2.6. We can find two paths from s to o and two paths from o to e. For the latter our notion of step distance causes no issues, as all paths between o and e have the same size. However, in the the case s to o, two paths of different length can be found. Hence, if one wants to consider all points at a certain step distance from o one obtains $\delta_1(s,o) = \delta_1(s, p_{23})$ but $p_{23} < o$. Allowing the conclusion that the successor is closer than its predecessor. This can be resolved by adjusting the density of the flow of time across* *branches, e.g. by manipulating the spacing between two points.*

A flow of time, in which this issue cannot arise is a flow where every path between two joins has the same length. Leading us to the definition of global-homogeneousness.

Definition 2.10. *Let $\Phi := \langle T, \leq, \omega \rangle$ be a quasi-metric flow of time. Then Φ is a global-homogeneous flow of time iff $\forall x, y, z \in T$*

$$\delta_\omega(x,y) + \delta_\omega(y,z) = \infty \vee \delta_\omega(x,z) = \delta_\omega(x,y) + \delta_\omega(y,z)$$

Notice, that every liner flow of time satisfies this property. Lastly, we have to check whether global homogeneousness resolves the issue.

Proposition 2.11. *Let $\Phi := \langle T, \leq, \omega \rangle$ flow of time then $\forall x, y, z \in T \; x < y < z \implies \delta_\omega(x,y) < \delta_\omega(x,z)$ if it is globally homogeneous*

Proof. If $\delta_\omega(x,y) + \delta_\omega(y,z) = \infty$, $\forall x \rightsquigarrow z \; y \notin x \rightsquigarrow z$. Hence, $x \leq y \leq z$. If either of $<$ are equal, we are done. Otherwise, we have $x < y < z$. Since, $\delta_\omega(x,z) = \delta_\omega(x,y) + \delta_\omega(y,z)$, thus $\delta_\omega(x,z) - \delta_\omega(y,z) = \delta_\omega(x,y)$ and since $\omega \in \mathbb{R}^+$, we know $\exists \varepsilon \, 0 < \varepsilon < \delta_\omega(y,z)$. Hence, $\delta_\omega(x,z) - \delta_\omega(y,z) < \delta_\omega(x,z) - \varepsilon < \delta_\omega(x,y)$. □

However, there is another notion of homogeneousness, namely local homogeneousness.

Example 2.12. *Consider the quasi-metric flow of time $\Phi := \langle T, <, \omega \rangle$ from Example 2.6. Interpret o as present. It is easy to see that the past is global-homogeneous, while the future is not. Even though, in the past there is a higher density of points in the bottom branch than in the top. While the inverse case holds in the future. So in some sense some strains of our time flow are denser than other.*

In Example 2.12 we regard "density" of time as the spacing of observations (Zumbach and Müller, 2001). In an analogue to real world problems, consider a sequence of graphs $(G_t)_{t \in T}$ distributed unevenly along a time line, as resulted from inconsistent measurements or a contraction of measurements into one order to obtain certain properties. Hence, this is a problem that can occure even in linear flows of time. Hence, we introduce the concept of local-homogeneousness, to express that all points in T are evenly spaced with respect to δ_ω. That is, we use a notion of homogeneity mentioned in (Zumbach and Müller, 2001).

Definition 2.13. *$\Phi := \langle T, \leq, \omega \rangle$ is a local-homogeneous flow of time iff*

$$\exists \varepsilon \forall x \in T \forall y \in s(x) \; \delta(x,y)_\omega = \varepsilon$$

It is clear to see that this characterises our problem. Thus we arrive at

Definition 2.14. *$\Phi := \langle T, \leq, \omega \rangle$ is a homogeneous flow of time iff Φ is locally and globally homogeneous.*

Note, however, that those are not the only notions of homogeneity, consider for example (Burgess, 1979) and (Venema, 1998). By using homogeneity we can now safely, navigate through a flow of time.

Definition 2.15. *Let $\Phi := \langle T, \leq, \omega \rangle$ be a global-homogeneous flow of time, then for some point of origin $x \in T$ and some $t \in \mathbb{R}$ we have*

$$\Phi(x,t)_\omega = \begin{cases} \{p \mid \forall p \in T \delta_\omega(x,p) = |t|\} & 0 \leq t \neq \infty \\ \{p \mid \forall p \in T \delta_\omega(p,x) = |t|\} & 0 > t \neq -\infty \\ \varnothing & otw. \end{cases}$$

If this point of origin is clear it is suppressed $\Phi(t)_\omega$.

However, more importantly, since global-homogeneity forces the distance between two points to be unambiguous, we can understand t to be a point on a linear time line and $\Phi(t)$ as a function that maps into a set of possible worlds. By restricting oneself to homogeneous flows of time, one ensures that all worlds across all branches are lined up. Thus allowing for aggregation of those worlds into one.

Sometimes it is beneficial to collapse the set of possible worlds into a single aggregated world, while one can easily conceive several strategies for such aggregation, this highly depends on the properties of $\nu(t)$. When dealing with discrete objects, such as formulas or discrete mathematical structures, one natural approach is based on some notion of stability. Here, we borrow ideas from modal logic and non-monotonic reasoning (Van Ditmarsch et al., 2007; Venema, 1998; Burgess, 1979; Gottlob, 1992).

Definition 2.16. *Let $\Phi := \langle T, \leq, \omega, \nu \rangle$ be a homogeneous flow of time. We call $\nu(p_t^s) = \Phi(t)_\omega^s := \{\phi \mid \forall p \in \Phi(t)_\omega \, \nu(p) \models \phi\}$ a stable (cautiously collapsed) set of formulas $\phi \in \mathcal{L}$ of the collapsed point p_t^s Moreover, by collapsing order is preserved, i.e. $\forall p \in \Phi(t-1) \forall q \in \Phi(t+1) \; p \leq p_t^s \leq q$.*

That is, this notion of stability ensures that we only consider those facts about the world that hold in all considered possible worlds as to make sure that all conclusions are founded on a stable set of premises. Here, it is easy to preserve consistency. Unfortunately, the other case, namely the unstable or bravely collapsed case, is far less straight forward. A detailed discussion is beyond the scope of this paper. For a more detailed discussion of corresponding issues see, e.g., (Bochman, 2003; Eiter et al., 2009; Gelfond and Lifschitz, 1988). As a consequence, we will only consider the stable approach in general.

Example 2.17. *Consider a family of graphs $\mathcal{G} := (G_i)_{i \in I}$ then we call the structure*

$\mathcal{G}^s := \langle \bigcap^{\equiv}_{G \in \mathcal{G}} V(G), \bigcap^{\equiv}_{G \in \mathcal{G}} E(G) \rangle$ *stable and* $\mathcal{G}^u := \langle \bigcup^{\equiv}_{G \in \mathcal{G}} V(G), \bigcup^{\equiv}_{G \in \mathcal{G}} E(G) \rangle$ *unstable. Where the equivalence is provided as additional information, such that equivalent objects can be seen as one.*

Moreover, an alternative approach imposes a preference realtion onto our points in time, allowing to focus only on those worlds that are the most preferred. For more information about preferred models consider (Kraus et al., 1990; Bochman, 2003). If the information embedded into the points in time is not inherently discrete, or if there is no requirement to remain crisp (Sunitha and Mathew, 2013), alternative methods of aggregation can be used. One example would be the assignment of conditional probabilities to each world (Chow and Liu, 1968). Thereby, not only creating a preference relation, but also opening up the aggregation of all possible worlds into one "weighted average" world.

3 Temporal Networks

There are a number of papers discussing and formalising temporal networks as well as showing some of their properties e.g. (Kempe et al., 2002), (Tang et al., 2009), (Holme and Saramäki, 2012) , (Kostakos, 2009), (Casteigts et al., 2012), (Michail, 2016). Especially since they are used across various field, such as biology, distributed systems, social network analysis, logistics, neuroscience and others, the concept of a temporal network has multiple formalisation and names, e.g. temporal graphs, evolving graphs, time-varying graphs, time-aggregated graphs, time-stamped graphs, dynamic networks, dynamic graphs, dynamical graphs just to name a few (Holme and Saramäki, 2012; Casteigts et al., 2012).

As mentioned above the general approach of multilayer networks was introduced to unify different formalisms that extend the ordinary notion of a graph. This includes several formalisms concerned with capturing the notion of a temporal network.

Even with all this diversity of formalisms, a fundamental distinction between two kinds of temporal networks has to be discussed. In (Holme and Saramäki, 2012) the distinction between instance-based temporal networks, called contact sequences, and interval-based temporal networks, called interval graphs is made. The prior understands time as a sequence of instances, where for example an edge is labeled with a sequence of time stamps indicating the network instances in which it is present. This framework, has a discrete flavour, or at least a countable one. Thus it can easily be encoded as a multilayer network, i.e. a multiplex where each layer represents an instance in time populated by edges with the corresponding time stamps. The latter, approaches time from a more "continuous" perspective.

We focus our attention at the instance based contact sequences. Hence, within this paper we conflate the terms temporal networks and instance based temporal network.

Definition 3.1. *Let* $\mathcal{T} := \langle \mathcal{G}, \mathcal{R}, \equiv, \leq, \omega \rangle$ *a weighted multilayer network with equivalence and order. It is called a instant based temporal network such that*

- $\mathcal{G} := \{G_t \mid \forall t \in T\}$ *for some labelling* T.
- $\mathcal{R}$ *is a collection of all successor relations with respect to* $\leq$*, i.e.* $\mathcal{R} := \{R_{t_i t_j} \mid \forall G_{t_i}, G_{t_j} \in \mathcal{G} R_{t_i t_j} := \{(v,w) \mid \forall v \in G_{t_i} \forall w \in G_{t_j}\ S(v,w)\}\}$.
- ω *is defined such that it respects the intra-graph relation weights while assigning weights to every inter-graph relation, i.e.*

$$\omega_{t_i t_j} := \begin{cases} \omega_{t_i}(e) & t_i = t_j \\ \mathbb{R}^+ & t_i \neq t_j \wedge t_i \leq t_j \\ \infty & otw. \end{cases}$$

Moreover, one can easily observe that a temporal network can be understood as a flow of time with additional structure. That is, consider the flow in time $\mathcal{T} := \langle \mathcal{G}, \leq, \nu, \omega \rangle$. Now we fix ν to $\forall G_t \in \mathcal{G}\ \nu(G_t) = G_t$ where G_t is some kind of weighted graph $G_t := \langle V_t, E_t, \omega_t \rangle$. By fixing the world at a certain point in time to being the same as its label ν becomes redundant. Let $\equiv$ be in Definition **??** and let $\leq$ extend to the vertices as in Definition 1.4. Then $\mathcal{R}$ is just the set of successors with respect to our extended flow relation, where ω carries over. Hence, the discussion and notation in the previous section carries over.

3.1 Paths in Temporal Networks

The notion of a path in a static non-weighted graph is a fairly simple structure, with its length being defined as its size, i.e. the number of edges in the path. In this scenario the computation of the shortest path between two vertices, can be computed in an efficient manner due to the fact that this problem is a matroid. However, for our purposes even more significant, the notion of what "shortest" actually means is unambiguous (Tang et al., 2009; Wu et al., 2014). Unfortunately, this property is already in question when considering weighted graphs in general. Additionally, on a computational level, finding a shortest path becomes more complicated as one now has to deal with possibly negative cycles, i.e. a path where starting and endpoint are the same and the sum of its weights is smaller than 0.

Moreover, by attaching weights to an edge one implicitly attaches a certain semantic value to the corresponding number. In its most general form one can distinguish between measures of similarity, e.g. bonding strength between nodes or dissimilarity, e.g. distance between nodes. When analysing a network those two interpretations have to be distinguished carefully, as some measures, e.g. eigenvector centrality or degree centrality only provide sensible results when ω is a similarity measure, while others such as betweenness and closeness centrality require dissimilarity measures. Fortunately, it is possible to invert the semantic interpretation of the respective measure. One example of this would be $\omega_s(x,y) = \frac{1}{1+\omega_d(x,y)}$. Consisting of multiple weighted graphs and edges with weights connecting those graphs, each of which having (possibly) different semantics. The same holds true for temporal networks where we have a dissimilarity measure on inter-graph edges and another measure with different semantic on intra-graph eges. Hence, making the notion of what a shortest path may be even more difficult (Runkler, 2012; Goshtasby, 2012; Segarra and Ribeiro, 2016).

Within the context of temporal networks, we found two insightful papers discussing specifically the issue of distance and the various shortest path problems that arise when discussing temporal networks (Tang et al., 2009; Wu et al., 2014). (Wu et al., 2014) introduces a set of minimum temporal paths, consisting of earliest-arrival path, i.e. starting from a_0 find the path ending in the smallest $b \in \overline{b}$, the latest-departure path, i.e. what is the largest $a \in \overline{a}$, while still being able to reach $\overline{b}$; the fastest path, i.e. what is the path shortest path between $\overline{a}$ and $\overline{b}$ minimising the difference between ending time and starting time and shortest path, i.e. the path that is the shortest with respect to traversal time. In this paper, traversal time corresponds with distance in time. Hence, in this case the latter two notions converge. In (Tang et al., 2009) a temporal network is conceptualised as a sequence of graphs. However, by limiting the amount of hops within each static graph, they manage to encode some sense of time into each static graph. Moreover, it is not uncommon to make the distinction between the size of a path and its duration explicit (Holme and Saramäki, 2012; Michail, 2016; Holme, 2015; Casteigts et al., 2012). As an analogue to flows in time, this distinction roughly corresponds to $|\cdot|_1$ and $|\cdot|_\omega$. As already mentioned, when dealing with weighted temporal networks an additional dimension is introduced. Namely, we distinguish not only between temporal steps and temporal distance, but also between intra-level steps and intra-level distance. One approach would be to use some norm to collapse those two dimensions, i.e. temporal and intra-level, into a single one. However, motivated by those examples and the issue of similarity and dissimilarity, we resist the urge to compress those dimensions. As a small benefit, one can use the concept of a path in time together with its notion of distance, similar the notion of a path in an ordinary graph.

Definition 3.2. *Let $\mathcal{T} := \langle \mathcal{G}, \mathcal{R}, \equiv, \preceq, \omega \rangle$ be a temporal network and let $v_{t_i} \in G_{t_i}$ and $w_{t_j} \in G_{t_i}$ then the alternating sequence of regular paths and forward-paths in time $\mathcal{T} \supseteq (v_{t_i} \rightsquigarrow_{\mathcal{T}} w_{t_j}) := v_{t_i} \rightsquigarrow^{+}_{\Phi} v_{t_k} \rightsquigarrow_{\mathcal{G}_{t_k}} v_{t_k} \cdots \rightsquigarrow^{+}_{\Phi} v_{t_j} \rightsquigarrow_{\mathcal{G}_{t_j}} w_{t_j}$ is called the temporal path from v_{t_i} to w_{t_j}. We write $\lambda_\Phi(x \rightsquigarrow_{\mathcal{T}} y)$ is the minimal set of paths in time, while $\lambda_{\mathcal{G}}(x \rightsquigarrow_{\mathcal{T}} y)$ is the minimal set of intra-graph paths. (Again the subscript is dropped if unambiguous)*

By deliberately distinguishing between the kinds of paths within a temporal network, we allow for an easy separation of measures, i.e. inter- and intra-length and size.

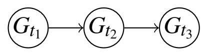
Definition 3.3. *Let $\mathcal{T} := \langle \mathcal{G}, \mathcal{R}, \equiv, \preceq, \omega \rangle$ be a temporal network, let $p := v_{t_i} \rightsquigarrow v_{t_j}$ be the temporal path from $v_{t_i} \in \mathcal{G}_{t_i}$ to $v_{t_j} \in {}_{t_i}$. Then we define*

$$|p|_\omega := (|E(\lambda_\Phi(p))|_\omega, |E(\lambda_{\mathcal{G}}(p))|_\omega)$$

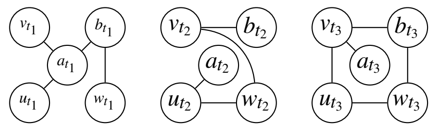
as the length of p with respect to ω and $|p|_1$ as its size.

By allowing for two dimensions with respect to path length and size we have to define a specific order on those values. A prime candidate for this is the so called product order, i.e. $(x_i, y_i) \leq (x_j, y_j) \iff x_i \leq x_j \wedge y_i \leq y_j$ (Dickson, 1913). However, one issue that immediately arises when using a product ordering, is the issue of partiality, i.e. we have multiple, possibly infinite, incomparable elements in our order as observed in Example 3.4.

Example 3.4. *The temporal networks $\mathcal{T}$*



Where G_{t_1}, G_{t_2} and G_{t_3} are respectively



Now consider the questions: What is a shortest path between a vertex in $\overline{a}$ and a vertex in $\overline{b}$? What is a shortest path between the vertex $\overline{a}_{t_i}$ and a vertex in $\overline{b}$? What is a shortest path between a vertex in $\overline{a}$ and the vertex $\overline{b}_{t_i}$? What is a shortest path between the vertex $\overline{a}_{t_i}$ and $\overline{b}_{t_j}$?

While each of those can give different results the more significant observation is that from the shortest path between $\overline{x}$ and $\overline{y}$, $\overline{x} \rightsquigarrow \overline{z} \rightsquigarrow \overline{y}$ it is not possible to conclude that $\overline{x} \rightsquigarrow \overline{z}$ is the shortest path between $\overline{x}$ and $\overline{z}$. Consider, $\overline{a} \rightsquigarrow \overline{w}$ and $\overline{a} \rightsquigarrow \overline{b}$. Thereby, prohibiting the save use of Dijkstra's algorithm for some of the specified problems.

$\lvert \cdot \rightsquigarrow \cdot \rvert$	$\overline{b}_{t_1}$	$\overline{b}_{t_2}$	$\overline{b}_{t_3}$	$\overline{b}$
$\overline{a}_{t_1}$	$(0,1)$	$(1,1)$	$(2,1)$	$(0,1)$
$\overline{a}_{t_2}$	(∞,∞)	$(0,4)$	$(1,2)$	$(0,4),(1,2)$
$\overline{a}_{t_3}$	(∞,∞)	(∞,∞)	$(0,2)$	$(0,2)$
$\overline{a}$	$(0,1)$	$(1,1),(0,4)$	$(2,1),(0,2)$	$(0,1)$

Table 1: Compute all possible shortest paths between $\overline{a}$ and $\overline{b}$ and all elements within those equivalence classes.

As presented in Example 3.4, there can be various natural interpretations of shortest path problems. For example, the problem $\overline{a} \rightsquigarrow_{\mathcal{T}} \overline{b}$ addresses the desire to compute the set of overall shortest distances between two equivalence classes, while the problem $\overline{a}_{t_i} \rightsquigarrow_{\mathcal{T}} \overline{b}$ restricts the same question to a certain starting point and $\overline{a} \rightsquigarrow_{\mathcal{T}} \overline{b}_{t_j}$ to a specific arrival date. Lastly, $\overline{a}_{t_i} \rightsquigarrow_{\mathcal{T}} \overline{b}_{t_j}$ is defacto a normal shortest path problem. Unfortunately, the definition of a multivalued distance measure induces several problems. Especially, frightening are some consequences w.r.t. computational complexity. Namely, in general multi-objective shortest path problems may need exponential runtime (Tarapata, 2007). Fortunately, due to its unique structure we can do better. This leads us to certain observations.

Proposition 3.5. *Let $\mathcal{T} := \langle \mathcal{G}, \mathcal{R}, \equiv, \leq, \omega \rangle$ be a global-homogeneous, temporal multiplex then for $v, w \in V(\mathcal{T})$ the length (and size) of all shortest path is comparable and thus the same.*

Proof. We know $\exists G_{t_i}, G_{t_j} \in \mathcal{G}\ v \in G_{t_i} \wedge w \in G_{t_j}$. If there does not exits a path between v and w we are done. Otherwise, by global-homogeneity and due to $\leq$ being directed, we obtain $\forall v \rightsquigarrow w\ |v \rightsquigarrow w|_\omega = (\delta_\omega(G_{t_i}, G_{t_j}), x)$. As they only differ in x all of them are comparable and by minimality all shortest paths have the same length. □

Proposition 3.6. *Let $\mathcal{T} := \langle \mathcal{G}, \mathcal{R}, \equiv, \leq, \omega \rangle$ be a global-homogeneous, temporal multiplex. The problem of finding the shortest path with respect to $|\cdot|_\omega \mathcal{T}$ from s to t for $s, t \in V(\mathcal{T})$ can be reduced in polynomial time to the problem of finding the shortest path between s and t in the weighted directed graph.*

Proof. Consider the construction $D := \langle V(\mathcal{T}), E(\mathcal{T}), \omega \rangle$. This transformation can be done in linear time. Show for $p_T := v \rightsquigarrow_{\mathcal{T}} w \subseteq \mathcal{T}$ we have $|v \rightsquigarrow_{\mathcal{T}} w|_\omega$ is minimal $\iff$ for $p_D := v \rightsquigarrow_D w \subseteq D$ we have $|v \rightsquigarrow_D w|_\omega$ is minimal. We observe $|E(\lambda_\Phi(p_T))|_\omega = \delta_\omega(v, w)$. Moreover, every path between v and w only differ in $|E(\lambda_{\mathcal{G}}(p_T))|_\omega$. And since $|p_D|_\omega = \delta_\omega(v, w) + |E(\lambda_{\mathcal{G}}(p_T))|_\omega$, we obtain $|p_T|_\omega$ minimal must be equivalent with $|p_D|_\omega$ minimal. □

Hence, finding the shortest path between two distinct vertices in a temporal network can be solved by applying a variant of Dijkstra's algorithm and it is thus $O((|V(\mathcal{T})| + |E(\mathcal{T})|) \cdot log(|V(\mathcal{T})|))$ (Barbehenn, 1998). By using this knowledge we can implement algorithms for computing the other shortest path problems. Even a naive implentation, i.e. one that computes in all distances between s_t and every member of $\overline{t}$ is in $O(|V(\mathcal{T})| \cdot (|V(\mathcal{T})| + |E(\mathcal{T})|) \cdot log(|V(\mathcal{T})|)) = O(n \cdot (m+n) \cdot log(n))$. Hence, by searching for all minimal paths with respect to $|\cdot|_\omega$ is bounded by the same computational complexity, as finding mimimal elements is at most $O(n^2)$. By applying the same algorithm for all $s_t \in \overline{s}$ we obtain at most $O((m+n) \cdot n^2 \cdot log(n))$. This is far from being tight. Serving only as a rough estimate to show polynomial membership, to justify this approach form a computational complexity point of view. For more on shortest path computation and computational complexity see (Wu et al., 2014; Tang et al., 2009; Michail, 2016) and (Cormen et al., 2009). By utilising this definition of a path, one can now move towards adapting path-based centrality measures.

4 Discussion

We decided against using the classical notion of multiplex network as a basis for our approach and instead defined a tailored variant that explicitly considers our generalized notion of time. This allows us to classify a temporal path as a sub-network of the main temporal network, with the temporal path retaining the properties of being a temporal network itself.

As discussed in (Taylor et al., 2017), one can add isolated "ghost nodes" to each graph G_t to obtain the (classical) multiplex property. Unfortunately, this implies that for certain measures the existence of such ghost nodes must be accounted for. Thus, for the time being, we assume that there are no "ghost nodes" in $G \in \mathcal{G}$.

Secondly, $E(\mathcal{R})$ is chosen to be neither transitive nor symmetric, as we use some notion of walk to move within a temporal network. This is because the transitivity property would correspond to skipping an instance of time and (figuratively) jumping into the future, while the symmetry property would allow travelling back in time. Both are properties we explicitly exclude. Unfortunately, this implies that computing the standard eigenvector centrality based on the supracentrality matrix is in general not sensible (Segarra and Ribeiro, 2016). Hence, the approach presented in (Taylor et al., 2017) cannot be used for our purposes.

Note that it is possible to define ω such that one layer in $\mathcal{T}$ represents an interval of time rather than an instance (a single time value). For example, for $e \in \mathcal{R}_{t_i t_{i+1}}$ we let $\omega_{t_i t_{i+1}}(e)$ be the length of the time interval for the graphs represented by G_{t_i} and if $e \in \mathcal{G}_t$ we let $\omega_t(e)$ encode interactions that occurred during this interval, e.g. the sum of all interactions between two vertices (Kivelä et al., 2014).

4.1 Implications for Path-based Centrality Measures

A path-based centrality measure is a measure that considers different aspects of the paths that exist between the vertices in a network in order to determine vertices that are considered "important" from a certain point of view. A prominent example of such a centrality measure is betweenness.

As betweenness relies on counts of shortest paths its extension to a multivalued path length is straight forward. However, the main issue arises from considering the various different shortest path problems. Hence, we define:

Definition 4.1. *Let $\mathcal{T} := \langle \mathcal{G}, \mathcal{R}, \equiv, \preceq, \omega \rangle$ be a linear, temporal multiplex, then weighted betweenness centrality is defined as*

$$C_B^{\omega}(\overline{v}) = \sum_{\substack{\overline{w},\overline{w}' \in V(\mathcal{T}) \setminus \{\overline{v}\} \\ \overline{w} \neq \overline{w}'}} \frac{\sigma^{\omega}_{\overline{w}\,\overline{w}'}(\overline{v})}{\sigma^{\omega}_{\overline{w}\,\overline{w}'}}$$

Where $\sigma^{\omega}_{\overline{x}\,\overline{x}'}$ is the number of shortest paths from $\overline{x}$ to $\overline{x}'$ with respect to $|\cdot|^{\mathcal{T}}_{\omega}$ and $\sigma^{\omega}_{\overline{x}\,\overline{x}'}$ is the number of shortest paths from x to x' with respect to $|\cdot|^{\mathcal{T}}_{\omega}$ that contain some element in $\overline{v}$. Moreover, in more general terms we write for $x \subseteq \overline{v}\; C^{\omega\, t_i}_{B\, t_j}(x)$ if we limit ourself to a subset of $\overline{v}$ where t_i and t_j (if specified) give an respective upper and lower bound for the layers to consider.

By only accounting for the number of shortest paths, we can neglect any form of spacing or issues of incomparability. Moreover, we actually obtain a total ordering among all vertices. Moreover, by considering $C_B^{\omega}(\overline{v})$ here the flow of time takes on a secondary position, as the centrality ranking of the vertices is based on their betweenness value at any given point in time. However, time remains an influential factor because for a vertex that connects two temporal layers the respective temporal betweenness value increases as well. Moreover, the bounded version of this measure is well suited to split the flow of time into a future and a present. That is, for a moment p we can compute $x \subseteq \overline{v}\; C_B^{\omega\, p}(x)$ to look into the past, and $x \subseteq \overline{v}\; C^{\omega}_{B\, p}(x)$ to compute the centrality based on the future.

5 Conclusion

Most approaches to the analysis of temporal networks do not explicitly discuss the underlying conception of time. Moreover, weighted temporal networks are still uncommon in the literature, and a direct discussion about how to reconcile the two semantic dimensions seems to be even more rare. In order to tackle those issues, this paper discussed time as a formal structure, thereby explicitly engaging with some of the underlying assumptions of time on which a temporal network may be operate.

Most prominently, we discuss some of the pitfalls that arise when dealing with non-deterministic time. Furthermore, our generalized abstraction of time promotes a clean separation of the semantics of time and the semantic interpretation of the network itself (i.e. the semantics of the vertices and edges in the corresponding network). By making this separation explicit, we provided a novel approach for analyzing temporal networks. Moreover, we introduced the notion of multivalued (temporal) paths that, on the one hand, enables a more in-depth understanding of temporal networks without sacrificing semantic integrity, while on the other hand introducing several new and interesting technical challenges, such as the computation of multivalued-centrality measures for temporal networks.

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